Write the answers to each **Group** in a separate answer-book.

2023

MATHEMATICS — HONOURS

Paper : DSCC-1

(Calculus, Geometry and Vector Analysis)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Calculus]

(Marks : 20)

1. Answer ***any two*** questions : 2×2

(a) If $f(x)=\begin{cases} \dfrac{x\left(e^{1/x}-e^{-1/x}\right)}{e^{1/x}+e^{-1/x}} & \text{when } x\neq 0 \\ 0, & \text{when } x=0 \end{cases}$,

show that $f(x)$ is not derivable at $x = 0$.

(b) If $x = \sin\theta$, $y = \sin p\theta$, then prove that $\left(1-x^2\right)y_2 - xy_1 + p^2 y = 0$.

(c) Find the length of a quadrant of the circle $r = 2a\sin\theta$.

2. Answer ***any four*** questions :

(a) Determine the value of p and q for which $\lim\limits_{x\to 0}\dfrac{x(1+p\cos x)-q\sin x}{x^3}$ exists and is equal to 1. 4

(b) If $\log_e y = \tan^{-1} x$, then show that

(i) $\left(1+x^2\right)y_2 + (2x-1)y_1 = 0$

(ii) $\left(1+x^2\right)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$. 2+2

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(c) Find a reduction formula for $I_{m,n}=\int \sin^m x \cos^n x\,dx$; where m, n are positive integers greater than 1.

Hence, find a reduction formula for $J_{m,n}=\int_0^{\pi/2} \sin^m x \cos^n x\,dx.$ 4

(d) Find the total length of the curve $r = a\cos\theta$. 4

(e) Find the surface area of the ellipsoid formed by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its major axis. 4

(f) The area enclosed by the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is revolved about X-axis. Find the volume of the solid generated. 4

Group - B

[Geometry]

(Marks : 35)

3. Answer ***any two*** questions : 2½×2

(a) In any conic, prove that the sum of the reciprocals of two perpendicular focal chords is constant.

(b) Find the angle of rotation of the axes about the origin which transforms the equation $x^2 - y^2 = 4$ to $x'y' = 2$.

(c) Find the equation of the sphere having the circle $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 5$ for a great circle.

4. Answer ***any five*** questions :

(a) Reduce the equation $5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0$ to its canonical form. Hence find the nature of the conic. 6

(b) Two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$ and the other is a tangent to $x^2 = 4by$. Show that the locus of the point of intersection of the lines is the curve

$(x^2 + y^2)(ax + by) + (bx - ay)^2 = 0.$ 6

(c) Show that the straight line $r\cos(\theta - \alpha) = p$ touches the conic $\frac{l}{r}=1+e\cos\theta$ if

$$(l\cos\alpha - ep)^2 + l^2\sin^2\alpha = p^2.$$

Also find the equation of the normal to the curve at the point of contact. 3+3

(d) A sphere of radius k passes through the origin O and meets the axes at A, B, C. Prove that the locus of the foot of the perpendicular to the plane ABC from O is given by $\left(x^2+y^2+z^2\right)^2\left(x^{-2}+y^{-2}+z^{-2}\right)=4k^2.$ 6

(e) A sphere S has points (0, 1, 0) and (3, –5, 2) at opposite ends of a diameter. Find the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the given sphere S is a great circle. 6

(f) Prove that the equation of the cylinder whose generators touch the sphere $x^2 + y^2 + z^2 = 4$ and are perpendicular to the plane $x + y - 2z = 0$ is $5x^2 + 5y^2 + 2z^2 + 4yz + 4zx - 2xy = 24$. 6

(g) Find the equations to generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$, which pass through the point (1, 1, 1). Hence, find the angle between these generators. 6

(h) Reduce the equation $7x^2 + y^2 + z^2 + 16yz + 8zx - 8xy + 2x + 4y - 40z - 14 = 0$ to the canonical form and find the nature of the conicoid it represents. 5+1

Group - C

[Vector Analysis]

(Marks : 20)

5. Answer ***any two*** questions : 2×2

(a) Prove that $\left[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\right]=\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$ for any three vectors $\vec{a},\vec{b},\vec{c}$.

(b) If $\vec{r}=t^2\hat{i}-e^{\sin\left(t\pi/2\right)}\hat{j}$ for $t\in[-1,1]$, then find $\int_{-1}^{1}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt$.

(c) Find the directional derivatives of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i}-\hat{j}-2\hat{k}$.

6. Answer ***any four*** questions :

(a) Show that three non-coplanar vectors $\vec{a},\vec{b},\vec{c}$ can be put as

$$\vec{a}=\frac{\vec{a}\cdot\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left(\vec{b}\times\vec{c}\right)+\frac{\vec{a}\cdot\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left(\vec{c}\times\vec{a}\right)+\frac{\vec{a}\cdot\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left(\vec{a}\times\vec{b}\right)$$

and similar expressions for $\vec{b}$ and $\vec{c}$. 4

(b) Find the shortest distance between the straight lines $\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+t\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$ and $\vec{r}=\left(k\hat{i}+3\hat{j}+4\hat{k}\right)+s\left(3\hat{i}+4\hat{j}+5\hat{k}\right)$. Find the vector equation of the plane containing any one of the given lines and the line of shortest distance. 2+2

(c) For any two vectors $\vec{u}$ and $\vec{v}$, prove that $\left(1+|\vec{u}|^2\right)\left(1+|\vec{v}|^2\right)=\left(1-\vec{u}\cdot\vec{v}\right)^2+\left|\vec{u}+\vec{v}+\left(\vec{u}\times\vec{v}\right)\right|^2$. 4

(d) Solve for $\vec{r}$ from the vector equation $k\vec{r}+\vec{a}\times\vec{r}=\vec{b},\ k\neq 0.$ 4

(e) If the vectors $a\hat{i}+\hat{j}+\hat{k},\ \hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k},(a,b,c\neq 1)$ are coplanar, then find the value of the expression $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}.$ 4

(f) A force $3\hat{i}+2\hat{j}-\hat{k}$ form a couple by acting at the two points (1, 0, 2) and (2, 1, 1). Find the moment of the couple, magnitude of the moment and hence find the perpendicular distance between the two parallel forces of the couple. 1+1+2

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