2024

MATHEMATICS — HONOURS

Paper : DSCC - 3

(Real Analysis)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ denote the set of all natural, real and rational numbers.

Group - A

(Marks : 30)

1. Answer ***any three*** questions :

(a) State the least upper bound axiom in $\mathbb{R}$. Prove that every non-empty subset of $\mathbb{R}$, which is bounded below has the greatest lower bound. 1+2

(b) Show that the set of all rational numbers is countable. 3

(c) Let $S=\left\{1,-1,1\frac{1}{2},-1\frac{1}{2},1\frac{1}{3},-1\frac{1}{3},\ldots\right\}$. Show that S is closed but not open. 3

(d) Find the derived set of the set $\left\{x\in\mathbb{R}:\sin\frac{1}{x}=0\right\}$. 3

(e) Prove or disprove : Every bounded infinite subset of $\mathbb{R}$ has an interior point. 3

2. Answer ***any three*** questions :

(a) State and prove Archimedean property of real numbers. Using Archimedean property show that $\inf\left\{\frac{1}{n}:n\in\mathbb{N}\right\}=0$. (1+4)+2

(b) (i) Let a and b be two real numbers such that $a<b$. Show that there is a rational number q and an irrational number r such that $a<q<b$; $a<r<b$.

(ii) Show that every infinite set has a countable infinite subset. (3+2)+2

(c) (i) Show that arbitrary union of open sets is an open set. Does the conclusion remain valid for arbitrary intersection of open sets? Justify.

(ii) Prove that an open interval is an open set. Is the converse true? Justify. (3+1)+(2+1)

(d) (i) Let S be a bounded set of real numbers and M be the lub of S. If $M \notin S$, show that M is a limit point of S. 3+4

(ii) Prove that the derived set of a set of real numbers is closed.

(e) State and prove Bolzano-Weierstrass Theorem on limit points. Show that every uncountable set has a limit point. (1+4)+2

Group - B

(Marks : 35)

3. Answer ***any four*** questions : 2×4

(a) Prove or disprove : $\left\{\frac{2^n}{n!}+\left(\frac{3}{5}\right)^n\right\}$ is a convergent sequence.

(b) Find $\limsup x_n$ and $\liminf x_n$, where $x_n = 2+(-1)^n$, $n \in \mathbb{N}$.

(c) Prove or disprove : If $\{x_n y_n\}$ is convergent then both $\{x_n\}$ and $\{y_n\}$ are convergent.

(d) Prove or disprove : $\left\{\frac{(-1)^n}{n}\right\}$ is a Cauchy sequence.

(e) Show that $\lim_{n\to\infty}\frac{x^n}{n!}=0$ for any $x \in \mathbb{R}$.

(f) Give example of a sequence which has exactly three subsequential limits.

4. Answer ***any three*** questions :

(a) Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences of non-zero real numbers. If $\lim_{n\to\infty} y_n \neq 0$, then prove that $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)=\frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$. 4

(b) State Sandwich Theorem. Prove that $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2-1}}+\ldots+\frac{1}{\sqrt{n^2-n}}\right)=1$. 1+3

(c) State and prove Cauchy's first limit theorem. 1+3

(d) Define Cauchy sequence. Prove that every Cauchy sequence is bounded. Is the converse true? Justify your answer. 1+2+1

(e) Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences. Prove that $\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim}(x_n + y_n)$. Cite an example for which the strict inequality holds. 3+1

5. Answer ***any three*** questions :

(a) A sequence $\{x_n\}$ is defined as follows :

$x_2 \le x_4 \le x_6 \le \ldots \le x_5 \le x_3 \le x_1$ and $\{y_n\}$ be defined by $y_n = x_{2n-1} - x_{2n}$ such that $y_n \to 0$ as $n \to \infty$. Show that the sequence $\{x_n\}$ is convergent. 5

(b) Prove that every sequence has a monotone subsequence. 5

(c) Let $\{[a_n, b_n]\}$ be a sequence of closed intervals such that each interval is contained in the preceeding one. Prove that there exists at least one point ξ such that $\xi \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$. If moreover $\lim_{n\to\infty} (b_n - a_n) = 0$, show that ξ is unique. If instead of sequence of closed intervals, sequence of open intervals be taken, will the result be true? Justify your answer. 3+1+1

(d) (i) Prove that $\lim_{n\to\infty} n^{1/n} = 1$.

(ii) Prove or disprove : For any two sequences $\{x_n\}$, $\{y_n\}$ of real numbers, $\limsup(x_n y_n) = \limsup(x_n) . \limsup(y_n)$. 3+2

(e) A sequence $\{x_n\}$ is defined by $x_1 = \sqrt{7}$, $x_{n+1} = \sqrt{x_n + 7}$ for all $n \ge 1$. Show that the sequence is convergent and converges to the positive root of $x^2 - x - 7 = 0$. 3+2

Group - C

(Marks : 10)

6. Answer ***any two*** questions :

(a) (i) Test the convergence of the following series :

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 1}$$

(ii) If $\sum_n x_n$ is a convergent infinite series of positive real numbers, prove that $\sum_n \frac{x_n}{1 + x_n}$ is convergent. 3+2

(b) Let $\sum u_n$ be a series of positive terms and $\limsup\, (u_n)^{\frac{1}{n}} = r$. Prove that $\sum u_n$ is convergent if $r < 1$ and $\sum u_n$ is divergent if $r > 1$. Hence show that $\sum_{n=1}^{\infty} \frac{1}{(n+1)^n}$ is convergent. 4+1

(c) State Gauss' test. Examine the convergence of the series $1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\ldots$. 1+4

(d) State and prove Leibnitz's test. 1+4
