2025

PHYSICS — HONOURS

Paper : DSCC-7

(Mathematical Physics - II)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question no.*** **1** and ***any five*** questions from the rest.

1. Answser ***any five*** questions :

(a) $y=x$ is a solution of the equation $\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$. Find the other solution. 3

(b) Show that two matrices related by similarity transformation have same eigenvalues. 3

(c) Under a coordinate transformation

$$x_i'=\sum_j a_{ij}x_j,$$

the length of a vector remains invariant. Find the nature of the matrix $A\equiv(a_{ij})$. 3

(d) Find the Jacobian of the coordinate transformation from cartesian to spherical polar coordinates. 3

(e) Three vectors $\vec{a},\vec{b},\vec{c}$ are linearly independent in ordinary three dimensional space of vectors. Find the condition(s) on them. 3

(f) For the matrix $H=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

Find its eigenvalues and eigenvectors. 1+2

(g) Let $|0\rangle$ and $|1\rangle$ be two orthonormal vectors. Two projection operators are defined as $P_0=|0\rangle\langle 0|$ and $P_1=|1\rangle\langle 1|$.

(i) Find P_0^2

(ii) For $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ find $P_0|\psi\rangle, P_1|\psi\rangle$. 1+2

(h) State the CFL condition of stability of the 1D wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}.$$

Mention the type of PDE where generally CFL is used and also one of the limitations of the method. 1+1+1

2. (a) Given the power series solution of the form $y = \sum_{k=0}^{\infty} a_k x^k$ of the equation $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$, find the recurrence relation. Show that for n = positive integer one solution is always a polynomial.

(b) Given the generating function for Legendre polynomial

$$G(x,h) = (1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(x)h^l$$

Show that

(i) $P_l(0) = 1$ for any l

(ii) $(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)$

(iii) $lP_l(x) = xP_l'(x) - P_{l-1}'(x)$, where $'$ denotes differentiation w.r.t. x. (4+2)+(2×3)

3. (a) For a differential equation of the form $y'' + p(x)y' + q(x)y = 0$, what is the condition for e^x to be a solution? What would be the condition for x to be the other solution? Using this information write an equation of the form $y'' + p(x)\, y'(x) + q(x)y = 0$ whose one solution is x and the other solution is e^x.

(b) For this last differential equation, determine the nature of the points (i) $x = 0$, (ii) $x = 1$.

(c) Show that x and e^x are linearly independent functions. (2+2+3)+(1+2)+2

4. (a) State and prove Cauchy-Schwarz inequality for a finite dimensional vector space.

(b) Given three vectors in $\mathbb{R}^3$(1, –1, 1), (–1, 0, 1) and (2, –1, 2), find whether they are linearly independent.

(c) Find an orthonormal set of vectors using Gram-Schmidt orthogonalisation procedure out of the vectors given in (b). 5+2+5

5. (a) A vector $|u\rangle = \begin{pmatrix} x \\ 3x \\ -2x \end{pmatrix}$, where '$x$' is an unknown real number. Find x such that $|u\rangle$ is normalized.

(b) Suppose $|u_1\rangle, |u_2\rangle, |u_3\rangle$ form an orthonormal basis. In this basis,

$$|\psi\rangle = 2i|u_1\rangle - 3|u_2\rangle + i|u_3\rangle \quad \text{and}$$
$$|\phi\rangle = 3|u_1\rangle - 2|u_2\rangle + 4|u_3\rangle$$

(i) Find $\langle\psi|$ and $\langle\phi|$.

(ii) If $a = 3i$, then find $\langle\phi|a\psi\rangle$ and $\langle a\phi|\psi\rangle$.

(c) Two vectors are orthogonal. Can we conclude from this statement that they are also linearly independent?

(d) Show that the eigenvalues of a Hermitian matrix are real. 2+(1+1+2+2)+2+2

6. (a) Consider a coordinate transformation $x_i' = \sum_{j=1}^{3} a_{ij} x_j$. Show that for two vectors A_i and B_j, $A_i B_j$ transform as a second rank tensor. What about $\sum_{i=1}^{3} A_i B_i$?

(b) We define $A_i = \sum_{j=1}^{3} S_{ij} B_j$, where B_j is a vector and S_{ij} is a second rank tensor. Show that A_i is a vector.

(c) Show that a symmetric 2nd rank tensor remains symmetric under the transformation law of tensors. (3+2)+4+3

7. (a) A be the matrix given by :

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$$

Find the eigenvalues of the matrix. Is there any degeneracy?

(b) Find the eigenvectors (normalised) for the eigenvalues.

(c) Find the unitary transformation that diagonalizes the matrix $\begin{pmatrix} 2 & -i \\ i & 2 \end{pmatrix}$. 3+5+4

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8. (a) If a matrix 'A' is Hermitian and $A^2 = \mathbb{I}$, show that A is also unitary.

(b) Find the trace and determinant of the matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Find the eigenvalues of this matrix using these values.

(c) Show that $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ anticommutes with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(d) A normal matrix N is defined as $NN^+ = N^+N$. If $N = A + iB$, where A and B are Hermitian then show that $[A, B] = 0$.

(e) Consider the transformation in three dimensions :

$x' = x \cos\theta + y \sin\theta \; ; \; y' = -x \sin\theta + y \cos\theta \; ; \; z' = z$

(i) Write down the transformation matrix $A(\theta)$.

(ii) Is $A(\theta)$ unitary? 2+3+2+2+(1+2)

9. (a) Deduce the standard five point formula for u where $\nabla^2 u = 0$ in 2D starting from the first principle of defining $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ etc. up to $O(h^2)$. Assume square mesh.

(b) Solve the 1D heat conduction problem $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions

$u(x, 0) = \sin\pi x$ for $0 \le x \le 1$

and $u(0, t) = u(1, t) = 0$

and $\alpha = 1$ in suitable units.

Perform *three* iterations and compare the value of $u(0.6, 0.04)$ with the exact solution, given by $u(x, t) = e^{-\pi^2 t} \sin\pi x$ giving rise to the exact value $u(0.6, 0.04) = 0.6408$. The formula of the finite difference method for solving this equation may be directly used. 6+(5+1)