2025

MATHEMATICS — HONOURS

Paper : DSCC-8

(Group Theory - II and Ring Theory - I)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

(*All symbols used have their usual meanings.*)

Group - A

[Group Theory - II]

(Marks : 40)

Answer ***any eight*** questions.

1. Let H be a subgroup of a group G such that $x^2 \in H$, for all $x \in G$. Show that H is a normal subgroup of G and G/H is commutative. 3+2

2. If ϕ be a homomorphism of a group G into a group G', show that

(i) $\phi(a^{-1}) = \{\phi(a)\}^{-1}$, for any $a \in G$

(ii) $\phi(a^n) = \{\phi(a)\}^n$, for any integer n and any $a \in G$. 2+3

3. State and prove Cayley's theorem for a group. 5

4. Let G be a group and H, K be subgroups of G such that G is an internal direct product of H and K. Show that

(i) $G \simeq H \times K$ and

(ii) $G/H \simeq K$. 3+2

5. Let G be a multiplicative group of order 10, and let $f : G \to G$ be a mapping defined by $f(x) = x^9$, $\forall x \in G$. Show that if G is commutative, then f is an automorphism. Is the converse true? Justify your answer. 3+2

6. (a) If G be a non-commutative group with centre $Z(G)$, then prove that $G/Z(G)$ is non-cyclic.

(b) Prove that a non-abelian group of order 10 must have a trivial centre. 3+2

7. Let G and G' be two groups and $\varphi : G \to G'$ be an epimorphism. Prove that $G/\text{Ker}\varphi$ is isomorphic to G'. 5

8. Let H, K be two finite cyclic groups of order m, n respectively. Prove that $H \times K$ is cyclic group if and only if $g.c.d.\ (m, n) = 1$. 5

9. (a) Let G be a cyclic group of order n. Prove that $|\text{Aut}(G)| = \phi(n)$, where ϕ is the Euler's phi function.

   (b) Let G be a group such that $\text{Aut}(G) = \{I_G\}$. Prove that G is a commutative group and $g^2 = e$, $\forall g \in G$. 3+2

10. Let G be a finite abelian group of order n and m be a positive divisor of n. Prove that G has a subgroup of order m. 5

11. Let H and K be two subgroups of a group G. If H is normal in G, prove that HK is a subgroup of G. Furthermore, if K is normal in G, then prove that HK is also normal in G. 5

Group - B

[Ring Theory - I]

(Marks : 35)

Answer ***any seven*** questions.

12. Show that a non-trivial finite ring having no divisor of zero is a ring with unity. 5

13. Define characteristic of a ring. Prove that the characteristic of an integral domain is either zero or prime number. 5

14. (a) Let a, b be two elements in a field F with $b \neq 0$. If $(ab)^2 = ab^2 + bab - b^2$, prove that $a = 1$.

    (b) Let R be an integral domain and $a \neq 0$ be an element of R. If $na = 0$ for some integer n, show that R is of finite characteristic. 3+2

15. Show that a finite integral domain is a field. 5

16. (a) Prove that $S = \{(a, 3b) : a, b \in Z\}$ is a subring of $Z \times Z$.

    (b) Give an example of a finite ring R with unity and a subring S of R having no unity with proper justification. 3+2

17. Define ideal of a ring. Let R be a ring and $a \in R$. Prove that $S = \{r \in R : r.a = 0\}$ is a left ideal of R. Give example to establish that it may not be a right ideal of R. 1+3+1

18. (a) For positive integer n, prove that the ring $\mathbb{Z}_n$ is isomorphic to the ring $\mathbb{Z}/n\mathbb{Z}$.

(b) For the ring homomorphism $f: \mathbb{Z}[x] \to \mathbb{Z}$ defined by $f(a_0 + a_1x + ... + a_nx^n) = a_0$, find Ker$f$. 3+2

19. Let R be a commutative ring with unity. Prove that an ideal I is maximal if and only if R/I is a field. 5

20. Prove that a commutative ring with unity has no non-trivial ideals if and only if it is a field. 5

21. (a) Let ρ be a congruence on a ring R. Prove that $I = \{r \in R \mid r \rho o\}$ is an ideal of R.

(b) Prove that $5\mathbb{Z}$ is a maximal ideal of the ring $\mathbb{Z}$. 3+2