2025

MATHEMATICS — HONOURS

Paper : DSCC-7

(Multivariate Calculus - I and Partial Differential Equations - I)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

All symbols have their usual meanings.

Group – A

[Multivariate Calculus – I]

(Marks : 60)

1. Answer ***any five*** questions : 2×5

(a) Find the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y^2}{x^2+y^2}$, if it exists.

(b) Find the equation of the tangent plane to the surface $z = \sin x + e^{xy} + 2y$ at the point (0, 1, 3).

(c) Verify whether the set $S = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ is an open set in $\mathbb{R}^2$ or not.

(d) Evaluate $\iint\limits_S (x^2+y^2)\,dS$, where $S: z = 4 - x - 2y,\ 0 \le x \le 4,\ 0 \le y \le 2$.

(e) If $x = u^2v^2$, $y = v^2 + u^2$, find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

(f) If $f = x^3 + y^3 + z^3$, find the directional derivative of f at the point (1, –1, 2) in the direction of the vector $\hat{j} - \hat{k}$.

(g) If $v = f(u)$, where u is a homogeneous function of two independent variables x, y of degree n, then prove that $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = nu\dfrac{dv}{du}$.

(h) If $f(t) = \int_0^t \sqrt{1+s^2}\,ds, s > 0$, then find $f'(2)$.

Please Turn Over

2. Answer *any five* questions :

(a) (i) Let (x, y) approach towards $(0, 0)$ along the line $y = -x$. Using Taylor's theorem for function of two variables, show that $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=-x)}} \dfrac{\sin xy + xe^x - y}{x\cos y + \sin 2y} = -2$.

(ii) Transform the equation $x^2 \dfrac{\partial z}{\partial x} + y^2 \dfrac{\partial z}{\partial y} = z^2$, by introducing new independent variable $u = x,\ v = \dfrac{1}{y} - \dfrac{1}{x}$ and the new function $w = \dfrac{1}{z} + \dfrac{1}{x}$. 5+5

(b) (i) Let $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Show that $f_{yx}(0, 0) \neq f_{xy}(0, 0)$. Which condition of Schwarz's theorem is not satisfied for this f?

(ii) State and prove Euler's theorem on homogeneous function of three variables. (4+1)+(1+4)

(c) (i) Use transformation $x = \dfrac{u}{v}, y = v$ to evaluate the integral $\iint\limits_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$.

(ii) Evaluate $\iiint\limits_E (9 - x^2 - y^2)dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$. 5+5

(d) (i) Find the area of the surface $z = \dfrac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$, in the region $0 \leq x \leq 1, 0 \leq y \leq 1$.

(ii) By changing the order of integration, prove that

$$\int_0^1 dx \int_x^{\frac{1}{x}} \frac{y\, dy}{(1+xy)^2(1+y^2)} = \frac{(\pi - 1)}{4}.$$ 5+5

(e) (i) If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, then prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 2u\left(1 - 4\sin^2 u\right)$.

(ii) Investigate the maxima and minima of the function $f(x, y) = x^2 + 4xy + 4y^2 + x^3 + 2x^2y + y^4$.

5+5

(f) (i) Assuming the validity of differentiation under the sign of integration, show that

$$\int_0^{\frac{\pi}{2}} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx = \pi\left(\sqrt{1+y} - 1\right), \text{ where } y > -1.$$

(ii) Evaluate $\iint \dfrac{dx\,dy}{(1+x^2+y^2)^2}$ over a loop of the lemniscate $(x^2+y^2)^2 - (x^2-y^2) = 0$. 5+5

(g) (i) Prove Taylor's theorem for a function $f(x, y)$ of two variables stating the conditions imposed upon $f(x, y)$.

(ii) Using the method of Lagrange's multiplier show that the minimum value of $bcx + cay + abz$ subject to $xyz = abc$ is $3abc$. 5+5

(h) (i) Let $f : D \to \mathbb{R}$ be a function where $D \subset \mathbb{R}^2$. When is a function $f(x, y)$ said to be differentiable at the point (a, b) in $D \subset \mathbb{R}^2$? Show that differentiability of f at (a, b) implies the continuity of f at (a, b) and existence of both f_x and f_y at that point.

(ii) What do you mean by an implicit function $y = \phi(x)$ defined by $F(x, y) = 0$ near (a, b)? Verify implicit function theorem for $x^2 + xy + y^2 - 1 = 0$ near $(0, -1)$. (1+2+2)+(2+3)

Group – B

[Partial Differential Equations – I]

(Marks : 15)

3. Answer ***any three*** questions :

(a) Solve the partial differential equation $\left(x^2 - yz\right)\dfrac{\partial z}{\partial x} + \left(y^2 - zx\right)\dfrac{\partial z}{\partial y} = z^2 - xy$ by Lagrange's method.

5

(b) Find the PDE arising from the family of surfaces $z = xy + f(x^2 + y^2)$, f being an arbitrary function. 5

(c) Find the complete integral of $p^2x + q^2y = z$ by Charpit's method, where $p \equiv \dfrac{\partial z}{\partial x}, q \equiv \dfrac{\partial z}{\partial y}$. 5

(d) Find the integral surface of the quasi linear PDE $yp - 2xyq = 2xz$ which passes through the curve $x = t$, $y = t^2$, $z = t^3$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. 5

(e) Obtain the solution of the Cauchy problem $xu_x + yu_y = u + 1$ with $u(x, y) = x^2$ on $y = x^2$. 5

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